Separability Properties of Three-mode Gaussian States

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We derive a necessary and sufficient condition for the separability of tripartite three mode Gaussian states, that is easy to check for any such state. We give a classification of the separability properties of those systems and show how to determine for any state to which class it belongs. We show that there exist genuinely tripartite bound entangled states and point out how to construct and prepare such states.

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I. INTRODUCTION

Entanglement of composite quantum systems is central to both the peculiarities and promises of quantum information. Consequently, the study of entanglement of bi- and multipartite systems has been the focus of research in quantum information theory. While pure state entanglement is fairly well understood, there are still many open questions related to the general case of mixed states. The furthest progress has been made in the study of systems of two qubits: it has been shown that a state of two qubits is separable if and only if its partial transpose is positive (PPT-property) [1] and a closed expression for the entanglement of formation was derived [2]. Moreover, it was shown [3] that all entangled states of two qubits can be *distilled* into maximally entangled pure states by local operations. This property of distillability is of great practical importance, since only the distillable states are useful for certain applications such as long-distance quantum communication, quantum teleportation or cryptography [4].

In higher dimensions much less is known: the PPT-property is no longer sufficient for separability as proved by the existence of PPT entangled states in $\mathbb{C}^2 \otimes \mathbb{C}^4$ systems [5]. These states were later shown to be *bound entangled* [6]: even if two parties (Alice and Bob) share an arbitrarily large supply of such states, they cannot transform ("distill") it into even a single pure entangled state by local quantum operations and classical communication. Meanwhile, a number of additional necessary or sufficient conditions for inseparability have been found for finite dimensional bipartite systems, which use properties of the range and kernel of the density matrix ρ and its partial transpose ρ^{T_A} to establish separability ([7] and references therein).

When going from two to more parties, current knowledge is even more limited. Pure multipartite entanglement was first considered in [8]. A classification of N-partite mixed states according to their separability properties has been given [9]. But even for three qubits there is currently no general way to decide to which of these classes a given state belongs [10]. Results on bound entanglement [11] and entanglement distillation [12] for multi-party systems have been obtained.

Recently increasing attention was paid to infinite dimensional systems, the so-called continuous quantum variables (CV), in particular since the experimental realization of CV quantum teleportation [13, 14]. Quantum information with CV in general is mainly concerned with the family of *Gaussian states*, since these comprise essentially all the experimentally realizable CV states. A practical advantage of CV systems is the relative ease with which entangled states can be generated in the lab [14, 15]. First results on separability and distillability of Gaussian states were reported in [16, 17, 18, 19, 20, 21, 22]. One finds striking similarities between the situations of two qubits and two one–mode CV systems in a Gaussian state: PPT is necessary and sufficient for separability [17, 18], and all inseparable states are distillable [19]. Generalizing the methods reviewed in [7] it was shown that for more than two modes at either side PPT entangled states exist [20]. In [21] a computable measure of entanglement for bipartite Gaussian states was derived.

The study of CV multipartite entanglement was initiated in [23, 24], where a scheme was suggested to create pure CV N-party entanglement using squeezed light and $N-1$ beamsplitters. In fact, this discussion indicates that tripartite entanglement has already been created (though not investigated or detected) in the CV quantum teleportation experiment [14].

In this paper we provide a complete classification of tri-mode entanglement (according to the scheme [9]) and obtain – in contrast to the finite dimensional case – a simple, directly computable criterion that allows to determine to which class a given state belongs. We show that none of these classes are empty and in particular provide examples of genuine tripartite bound entangled states, i.e. states of three modes A, B, and C that are separable whenever two parties are grouped together but cannot be written as a mixture of tripartite product states.

Before we can derive our results we need to introduce some notation and collect a number of useful facts about our main object of study: Gaussian states.

II. GAUSSIAN STATES

In quantum optics and in other scenarios described by continuous quantum variables, not all states on the infinite dimensional Hilbert space are equally accessible in current experiments. In fact, the set of *Gaussian states* comprises essentially all genuinely CV states that can currently be prepared in the lab. This, and the mathematical simplicity of these states are the reasons why CV quantum information has so far considered almost exclusively Gaussian states, as will the present paper. This section summarizes results on Gaussian states that we need in the following and introduces some notation.

We consider systems composed of n distinguishable infinite dimensional subsystems, each with Hilbert space $\mathcal{H} = \mathrm{L}^2(\mathbb{R})$. These could be implemented quantum optically by different modes of the electromagnetic field, hence each of these subsystems will be referred to as a "mode". To each mode belong the two canonical observables $X_k, P_k, k = 1, \ldots, n$ with commutation relation $[X_k, P_k] = i$. Defining $R_k = X_k, R_{n+k} = P_k$ these relation are summarized as $[R_k, R_l] = -iJ_{kl}$, using the antisymmetric $2n \times 2n$ matrix

$$J = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}. \tag{1}$$

which plays an important role in the following calculations [25].

For such systems, it is convenient to describe the state ρ by its characteristic function

$$\chi(x) = \mathrm{tr}[\rho D(x)]. \tag{2}$$

Here $x = (q, p)$, $q, p \in \mathbb{R}^n$ is a real vector, and

$$D(x) = e^{-i\sum_k (q_k X_k + p_k P_k)}. \tag{3}$$

The characteristic function contains all the information about the state of the system, that is, one can construct ρ knowing χ. Gaussian states are exactly those for which χ is a Gaussian function of the phase space coordinates x [26],

$$\chi(x) = e^{-\frac{1}{4}x^T\gamma x - id^T x}, \tag{4}$$

where γ is a real, symmetric, strictly positive matrix, the *correlation matrix* (CM), and $d \in \mathbb{R}^{2n}$ is a real vector, the *displacement*. Note that both γ and d are directly measurable quantities, their elements γ_{kl} and d_k are related to the expectation values and variances of the operators R_k. A Gaussian state is completely determined by γ and d. Note that the displacement of a (known) state can always be adjusted to $d = 0$ by a sequence of unitaries applied to individual modes. This implies that d is irrelevant for the study of nonlocal properties. Therefore we will occasionally say, e.g., that "a CM is separable" when the Gaussian state with this CM is separable. Also, from now on in this paper "state" will always mean "Gaussian state" (unless stated otherwise).

Not all real, symmetric, positive matrices γ correspond to the CM of a physical state. There are a number of equivalent ways to characterize physical CMs, which will all be useful in the following. We collect them in

Lemma 1 *(Correlation Matrices)*
For a real, symmetric $2n \times 2n$ matrix $\gamma > 0$ the following statements are equivalent:

$$\gamma \text{ is the CM of a physical state}, \tag{5a}$$

$$\gamma + J\gamma^{-1}J \geq 0, \tag{5b}$$

$$\gamma - iJ \geq 0, \tag{5c}$$

$$\gamma = S^T(D \oplus D)S, \tag{5d}$$

for S symplectic [27] and $D \geq \mathbb{1}$ diagonal [28].

Proof: $(5a) \Leftrightarrow (5b)$ see [26]; $(5a) \Leftrightarrow (5c)$ see [20]; $(5a) \Leftrightarrow (5d)$ see [29, Prop. 4.22].

A CM corresponds to a pure state if and only if (iff) $D = \mathbb{1}$, i.e. iff $\det\gamma = 1$ (e.g. [26]). It is easy to see from (5d) that for pure states Ineq. (5b) becomes an equality and $\dim[\ker(\gamma - iJ)] = n$. It is clear from Eq. (5d) that for every CM γ there exists a pure CM γ_0 such that $\gamma_0 \leq \gamma$. This will allow us to restrict many proofs to pure CMs only. Note that for a pure $2n \times 2n$ CM γ it holds that $\mathrm{tr}\gamma \geq 2n$.

A very important transformation for the study of entanglement is partial transposition [1]. Transposition is an example of a positive but not completely positive map and therefore may reveal entanglement when applied to part of an entangled system. On phase space transposition corresponds to the transformation that changes the sign of all the p coordinates $(q, p) \mapsto \Lambda(q, p) = (q, -p)$ [18] and leaves the q's unchanged. For γ and d this means $(\gamma, d) \mapsto (\Lambda\gamma\Lambda, \Lambda d)$. Using this, the NPT-criterion for inseparability [1] translates very nicely to Gaussian states. Consider a bipartite system consisting of m modes on Alice's side and n modes on Bob's ($m \times n$-system in the following). Let γ be the CM of a Gaussian $m \times n$-state and denote by $\Lambda_A = \Lambda \oplus \mathbb{1}$ the partial transposition in A's system only. Then we have the following criterion for inseparability:

Theorem 1 *(NPT criterion)*
Let γ be the CM of a $1 \times n$ system, then γ corresponds to a inseparable state if and only if $\Lambda_A\gamma\Lambda_A$ is not a physical CM, i.e. if and only if

$$\Lambda_A\gamma\Lambda_A \ngeq iJ. \tag{6}$$

We say that γ "is NPT" if (6) holds.

PROOF: See [18] for $N = 1$ and [20] for the general case.

Occasionally it is convenient to apply the orthogonal operation Λ_A to the right hand side of Ineq. (6) and write $\tilde{J}_A \equiv \Lambda_A J \Lambda_A$.

For states of at least two modes at both sides Condition (6) is still sufficient for inseparability, but no longer necessary as shown by Werner and Wolf, who have considered a family of 2×2 entangled states with positive partial transpose [20]. In the same paper, it was shown that

Theorem 2 *(Separability of Gaussian States)*
A state with CM γ is separable iff there exist CMs γ_A, γ_B such that

$$\gamma \geq \gamma_A \oplus \gamma_B. \tag{7}$$

It is observed in [20] that if Ineq. (7) can be fulfilled, then the state with CM γ can be obtained by local operations and classical communication from the product state with CM $\gamma_p = \gamma_A \oplus \gamma_B$, namely by mixing the states (γ_p, d) with the d's distributed according to the Gaussian distribution $\propto \exp\left[-d^T(\gamma - \gamma_p)^{-1}d\right]$.

Note that while Theorem 2 gives a necessary and sufficient condition for separability, it is not a practical criterion, since to use it, we have to prove the existence or nonexistence of CMs γ_A, γ_B. Instead, a criterion would allow to directly calculate from γ whether the corresponding state is separable or not. Theorem 2 and its extension to the 3-party situation are the starting point for the derivation of such a criterion for the case of three-mode three-party states in the following main section of this paper.

III. TRI-MODE ENTANGLEMENT

When systems that are composed of $N > 2$ parties are considered, there are many "types" of entanglement due to the many ways in which the different subsystems may be entangled with each other. We will use the scheme introduced in [9], to classify three-mode tripartite Gaussian states. The important point is that from the extension of Theorem 2 we can derive a simple criterion that allows to determine which class a given state belongs to. This is in contrast to the situation for three qubits, where up until now no such criterion is known. In particular, we show that none of these classes are empty and we provide an example of a genuine tripartite bound entangled state, i.e. a state of three modes A, B, and C that is separable whenever two parties are grouped together but cannot be written as a mixture of tripartite product states and therefore cannot be prepared by local operations and classical communication of three separate parties.

A. Classification

The scheme of [9] considers all possible ways to group the N parties into $m \leq N$ subsets, which are then themselves considered each as a single party. Now, it has to be determined whether the resulting m-party state can be written as a mixture of m-party product states. The complete record of the m-separability of all these states then characterizes the entanglement of the N-party state.

For tripartite systems, we need to consider four cases, namely the three bipartite cases in which AB, AC, or BC are grouped together, respectively, and the tripartite case in which all A, B, and C are separate. We formulate a simple extension to Theorem 2 to characterize mixtures of tripartite product states

Theorem 2' *(Three-party Separability)*
A Gaussian three-party state with CM γ can be written as a mixture of tripartite product states iff there exist one-mode correlation matrices $\gamma_A, \gamma_B, \gamma_C$ such that

$$\gamma - \gamma_A \oplus \gamma_B \oplus \gamma_C \geq 0. \tag{8}$$

Such a state will be called fully separable.
PROOF: The proof is in complete analogy with that of Theorem 7 in [20] and is therefore omitted here.

A state for which there are a one-mode CM γ_A and a two-mode CM γ_{BC} such that $\gamma - \gamma_A \oplus \gamma_{BC} \geq 0$ is called *$A-BC$ biseparable* (and similarly for the two other bipartite groupings). In total, we have the following five different entanglement classes:

Class 1 *Fully inseparable states* are those which are not separable for any grouping of the parties.

Class 2 *1-mode biseparable states* are those which are separable if two of the parties are grouped together, but inseparable with respect to the other groupings.

Class 3 *2-mode biseparable states* are separable with respect to two of the three bipartite splits but inseparable with respect to the third.

Class 4 *3-mode biseparable states* separable with respect to all three bipartite splits but cannot be written as a mixture of tripartite product states.

Class 5 The *fully separable* states can be written as a mixture of tripartite product states.

Examples for Class 1 (the GHZ-like states of [24]), Class 2 (two-mode squeezed vacuum in the first two and the vacuum in the third mode), and Class 5 (vacuum state in all three modes) are readily given; we will provide examples for Classes 3 and 4 in Subsection IV below.

How can we determine to which Class a given state with CM γ belongs? States belonging to Classes 1, 2, or 3 can be readily identified using the NPT-criterion (Theorem 1). Denoting the partially transposed CM by $\tilde{\gamma}_x = \Lambda_x \gamma \Lambda_x, x = A, B, C$, we have the following equivalences:

Lemma 2 *(Classification)*

$$\tilde{\gamma}_A \not\geq iJ, \tilde{\gamma}_B \not\geq iJ, \tilde{\gamma}_C \not\geq iJ \;\Leftrightarrow\; \textit{Class 1} \tag{9}$$

$$(*)\tilde{\gamma}_A \not\geq iJ, \tilde{\gamma}_B \not\geq iJ, \tilde{\gamma}_C \geq iJ \;\Leftrightarrow\; \textit{Class 2} \tag{10}$$

$$(*)\tilde{\gamma}_A \not\geq iJ, \tilde{\gamma}_B \geq iJ, \tilde{\gamma}_C \geq iJ \;\Leftrightarrow\; \textit{Class 3} \tag{11}$$

$$\tilde{\gamma}_A \geq iJ, \tilde{\gamma}_B \geq iJ, \tilde{\gamma}_C \geq iJ \;\Leftrightarrow\; \textit{Class 4 or 5}, \tag{12}$$

where the $(*)$ *reminds us to consider all permutations of the indices A, B, and C.*

The proof follows directly from the definitions of the different classes and Theorem 1.

What is still missing is an easy way to distinguish between Class 4 and Class 5. Thus to complete the classification we now provide a criterion to determine whether a CM γ satisfying Ineqs. (12) is fully separable or 3-mode biseparable, that is we have to decide whether there exist one-mode CMs $\gamma_A, \gamma_B, \gamma_C$ such that (8) holds, in which case γ is fully separable. In the next subsection we will describe a small set consisting of no more than nine CMs among which γ_A is necessarily found if the state is separable.

B. Criterion for Full Separability

This subsection contains the main result of the paper: a separability criterion for PPT $1 \times 1 \times 1$ Gaussian states, i.e. states whose CM fulfills Ineqs. (12). We start from Theorem 2' and obtain in several steps a simple, directly computable necessary and sufficient condition. The reader mainly interested in this result may go directly to Theorem 3, from where she will be guided to the necessary definitions and Lemmas.

Since the separability condition in Theorem 2' is formulated in terms of the positivity of certain matrices the following lemma will be very useful throughout the paper. We consider a self-adjoint $(n+m)\times(n+m)$ matrix M that we write in block form as

$$M = \begin{pmatrix} A & C \\ C^\dagger & B \end{pmatrix}, \tag{13}$$

where A, B, C are $n \times n, m \times m$, and $n \times m$ matrices, respectively.

Lemma 3 *(Positivity of self-adjoint matrices)*
A self-adjoint matrix M as in (13) with $A \geq 0, B \geq 0$ *is positive if and only if for all* $\epsilon > 0$

$$A - C\frac{1}{B+\epsilon\mathbb{1}}C^\dagger \geq 0, \tag{14}$$

or, equivalently, if and only if

$$\ker B \subseteq \ker C \tag{15a}$$

and

$$A - C\frac{1}{B}C^\dagger \geq 0, \tag{15b}$$

where B^{-1} *is understood in the sense of a pseudoinverse (inversion on the range).*

Proof: The only difficulty in the proof arises if $\ker B \neq 0$. Therefore we consider the matrices M_ϵ, where B in (13) is replaced by $B_\epsilon = B + \epsilon\mathbb{1}$ $(\epsilon > 0)$, which avoid this problem and which are positive $\forall\epsilon > 0$ iff $M \geq 0$. In a second simplifying step we note that $M_\epsilon \geq 0$ $\forall\epsilon > 0$ iff $M'_\epsilon = (\mathbb{1} \oplus B_\epsilon^{-1/2})M(\mathbb{1} \oplus B_\epsilon^{-1/2}) \geq 0$.

Now direct calculation shows the claim: we can write a general $f \oplus g$ as $f \oplus \left[(B_\epsilon^{-1/2}C^\dagger)h + h_\perp\right]$, where $h_\perp$ is orthogonal to the range of $(B_\epsilon^{-1/2}C^\dagger)$. Then $(f\oplus g)^\dagger M'_\epsilon(f\oplus g) = f^\dagger(A - CB_\epsilon^{-1}C^\dagger)f + (f+h)^\dagger CB_\epsilon^{-1}C^\dagger(f+h) + h_\perp^\dagger h_\perp$, which is clearly positive, if (14) holds. With the choice $h_\perp = 0$ and $h = -f$ it is seen that (14) is also necessary.

That the second condition is equivalent is seen as follows: If Ineq. (14) holds $\forall\epsilon > 0$ there cannot be vector $\xi \in \ker B$ and $\xi \notin \ker C$ since for such a ξ we have $\xi^T\left(A - C\frac{1}{B+\epsilon\mathbb{1}}C^\dagger\right)\xi < 0$ for sufficiently small $\epsilon > 0$, and if (15a) holds then (14) converges to (15b). Conversely, if (15a) holds, then $CB^{-1}C^\dagger$ is well-defined and Ineq. (15b) implies it $\forall\epsilon > 0$. ∎

As mentioned above, in this section we exclusively consider three-mode CMs γ that satisfy Ineqs. (12). We write γ in the form of Eq. (13) as

$$\gamma = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}, \tag{16}$$

where A is a 2×2 matrix, whereas B is a 4×4 matrix. We observe that Ineqs. (12) impose some conditions on γ that will be useful later on:

Observation 1 *Let* γ *satisfy Ineqs. (12), then*

$$\gamma \geq \begin{pmatrix} \sigma_A iJ & 0 & 0 \\ 0 & \sigma_B iJ & 0 \\ 0 & 0 & \sigma_C iJ \end{pmatrix}, \tag{17}$$

where $\sigma_x \in \{0, \pm 1\}, \forall x = A, B, C.$

Proof: Ineqs. (12) say that $\gamma \pm iJ \geq 0$ and $\gamma \pm i\tilde{J}_x \geq 0$ $\forall x$. By adding these positive matrices all combinations of σ_x can be obtained. ∎

From this it follows:

Observation 2 *For a PPT CM* γ *as in Eq. (16)*

$$\ker(B + iJ), \ker(B + i\tilde{J}) \subseteq \ker C, \tag{18}$$

where $\tilde{J} = J \oplus (-J)$ *is the partially transposed J for two modes.*

Proof: Cond. (18) on the kernels is an immediate consequence of Lemma 3 applied to the matrices $\gamma - 0 \oplus iJ \oplus (\pm iJ)$, which are positive by Obs. 1. ∎

Then the matrices

$$\tilde{N} \;\equiv\; A - C\frac{1}{B - i\tilde{J}}C^T, \tag{19a}$$

$$N \;\equiv\; A - C\frac{1}{B - iJ}C^T \tag{19b}$$

are well-defined and

Observation 3 *It holds that both*

$$\mathrm{tr}N, \mathrm{tr}\tilde{N} > 0, \tag{20}$$

PROOF: Cond. (20) is true since, again by Lemma 3 and Obs. 1, both N and $\tilde{N}$ are positive and $N \pm iJ, \tilde{N} \pm iJ \geq 0$. This implies that $N, \tilde{N}$ cannot be zero, which is the only positive matrix with vanishing trace. Therefore $\mathrm{tr}N, \mathrm{tr}\tilde{N}$ are strictly positive. ∎

The remainder of this section leads in several steps to the separability criterion. First, we simplify the condition (8) by reducing it to a condition which involves only one one-mode CM γ_A.

Lemma 4 *A PPT 3-mode CM γ is fully separable if and only if there exists a one-mode CM γ_A such that*

$$\tilde{N} \;\geq\; \gamma_A, \tag{21a}$$
$$N \;\geq\; \gamma_A, \tag{21b}$$

where $N, \tilde{N}$ were defined in Eqs. (19). Without loss of generality we require γ_A to be a pure state CM, i.e. $\det \gamma_A = 1$.

PROOF: By Theorem 2' full separability of γ is equivalent to the existence of one-mode CMs $\gamma_A, \gamma_B, \gamma_C \geq iJ$ such that $\gamma - \gamma_A \oplus \gamma_B \oplus \gamma_C \geq 0$. Let γ_x stand for $\gamma_{A,B,C}$.

By Lemma 3 this is equivalent to $\exists \gamma_x$ such that $X_\epsilon \equiv B - C^T \frac{1}{A_\epsilon - \gamma_A} C \geq \gamma_B \oplus \gamma_C$, $\forall \epsilon > 0$, where $A_\epsilon \equiv A + \epsilon \mathbb{1}$. But iff there exist such γ_x then (Lemma 3) the inequality also holds for $\epsilon = 0$ and the kernels fulfill (15a). This is true iff the matrix $X \equiv X_0'$ is a CM belonging to a separable state, i.e. (Theorem 1) iff $X' \geq i\tilde{J}, iJ$. Using $B \geq i\tilde{J}, iJ$ [which holds since γ fulfills Ineqs. (12)] we obtain that γ is separable iff there exists $\gamma_A \geq iJ$ such that

$$\begin{pmatrix} A - \gamma_A & C \\ C^T & B_k' \end{pmatrix} \geq 0, \; k = 1, 2, \tag{22}$$

where $B_1' = B - iJ$ and $B_2' = B - i\tilde{J}$. Since Condition (15a) holds, this is (Lemma 3) equivalent to Ineqs. (21). That we can always choose $\det \gamma_A = 1$ follows directly from Eq. (5d) and the remark after Lemma 1. ∎

While we can always find a γ_A fulfilling Ineq. (21b), since γ belongs to a PPT state (and there exists a two-mode CM $\gamma_{BC} \geq iJ$ such that $\gamma_A \oplus \gamma_{BC}$ is smaller than γ), it may well happen that Ineq. (21a) cannot be satisfied at all, or that it is impossible to have both Ineqs. (21) fulfilled for one γ_A simultaneously. Note that due to Ineqs. (12), N and $\tilde{N}$ as above are always positive. From Ineqs. (21) we observe that

Observation 4 *for the CM γ of a separable state it is necessary to have*

$$\mathrm{tr}N, \mathrm{tr}\tilde{N} \;\geq\; 2, \tag{23a}$$
$$\det N, \det \tilde{N} \;>\; 0, \tag{23b}$$

where γ as in Eq. (16) and $N, \tilde{N}$ as in Eqs. (19).

PROOF: A self-adjoint 2×2 matrix is positive iff its trace and determinant are positive. Since the trace of the RHS of both Ineqs. (21) is ≥ 2 [remark after Lemma 1] the same is necessary for the LHS. Also, since $\det \gamma_A = 1$, which implies that γ_A has full rank, any matrix $\geq \gamma_A$ must also have full rank [31] and thus a strictly positive determinant. ∎

For a self-adjoint positive 2×2 matrix

$$R = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}, \tag{24}$$

we show

Lemma 5 *There exists a CM $\gamma_A \leq R$ if and only if there exist $(y, z) \in \mathbb{R}^2$ such that*

$$\mathrm{tr}R \;\geq\; 2\sqrt{1 + y^2 + z^2}, \tag{25a}$$
$$\det R + 1 + L^T \begin{pmatrix} y \\ z \end{pmatrix} \;\geq\; \mathrm{tr}R \sqrt{1 + y^2 + z^2}, \tag{25b}$$

where

$$L = (a - c, 2\mathrm{Re}b). \tag{26}$$

PROOF: As noted in Lemma 4 we need only look for γ_A with $\det \gamma_A = 1$. We parameterize

$$\gamma_A = \begin{pmatrix} x + y & z \\ z & x - y \end{pmatrix}, \tag{27}$$

with real parameters x, y, z and $x^2 = 1 + y^2 + z^2$ for purity. This is a CM iff $\gamma_A - iJ \geq 0$ (Lemma 1), that is iff $\mathrm{tr}\gamma_A = 2x \geq 0$ (where we use that positivity of the a 2×2 matrix is equivalent to the positivity of its trace and determinant and $\det(\gamma_A - iJ) = 0$ by construction). By the same argument, $R - \gamma_A \geq 0$ leads to the two conditions (25). ∎

The inequalities (25) have a simple geometrical interpretation that will be useful for the proof of the promised criterion: Ineq. (25a) restricts (y, z) to a circular disk $\mathcal{C}'$ of radius $\sqrt{(\mathrm{tr}R)^2/4 - 1}$ around the origin, while Ineq. (25b) describes a (potentially degenerate) ellipse $\mathcal{E}$ (see Fig. 2), whose elements are calculated below, and the existence of a joint solution to Ineqs. (25) is therefore equivalent to a nonempty intersection of $\mathcal{C}'$ and $\mathcal{E}$.

Applying this now to the matrices (19) we find that in order to simultaneously satisfy both conditions in Lemma 4, the intersection between the two ellipses $\mathcal{E}, \tilde{\mathcal{E}}$ and the smaller of the two concentric circles $\mathcal{C}', \tilde{\mathcal{C}}'$ (which we denote in the following by $\mathcal{C}$) must be nonempty. This condition leads to three inequalities in the coefficients of the matrices $\tilde{N}, N$ which can be satisfied simultaneously if and only if the PPT trimode state is separable. Thus we can reformulate the condition for separability (Lemma 4) as follows

Lemma 6 *(Reformulated Separability Condition) A three-mode state with CM γ satisfying Ineqs. (12) is*

fully separable if and only if there exists a point $(y,z)\in\mathbb{R}^2$ *fulfilling the following inequalities:*

$$\min\{\mathrm{tr}N,\mathrm{tr}\tilde{N}\} \;\geq\; 2\sqrt{1+y^2+z^2}, \tag{28a}$$

$$\det N+1+L^T\begin{pmatrix}y\\z\end{pmatrix} \;\geq\; \mathrm{tr}N\sqrt{1+y^2+z^2}, \tag{28b}$$

$$\det \tilde{N}+1+\tilde{L}^T\begin{pmatrix}y\\z\end{pmatrix} \;\geq\; \mathrm{tr}\tilde{N}\sqrt{1+y^2+z^2}. \tag{28c}$$

PROOF: According to Lemma 4 γ belongs to a separable state iff we can find γ_A smaller than $\tilde{N}$ and smaller than N. According to Lemma 5 we can find such a γ_A iff we can find (y,z) such that Ineqs. (25) are satisfied for both N and $\tilde{N}$. ∎

In the following paragraphs we have a closer look at the sets $\mathcal{E},\tilde{\mathcal{E}}$, and $\mathcal{C}$. The goal of this discussion is to identify a few special points – directly computable from γ – among which a solution to Ineqs. (28) will be found iff the state under consideration is separable. This will then lead to the final practical form of the separability criterion which is stated at the end of this section.

By squaring Ineq. (28b) we obtain

$$\left[\begin{pmatrix}y\\z\end{pmatrix}-\mu L\right]^T K\left[\begin{pmatrix}y\\z\end{pmatrix}-\mu L\right]\leq m, \tag{29}$$

where $\mu=(\det N+1)/k_1$, $m=\frac{k_2}{k_1}\left[(\det N+1)^2-k_1\right]$, and the matrix K is [30]

$$K=k_1P_L+k_2P_{L^\perp},$$

with the orthogonal projectors $P_L,P_{L^\perp}$ on $L,L^\perp$ and

$$\begin{aligned}k_1 &= 4\left[\det N+(\mathrm{Im}b)^2\right],\\ k_2 &= (\mathrm{tr}N)^2.\end{aligned}$$

Due to Ineqs. (23) k_1 and k_2 are strictly positive, μ,m are well-defined and K is a positive matrix of rank 2. Let us now distinguish the cases $m<0$ and $m\geq 0$. For $m<0$ Ineq. (29) can never be fulfilled since K is a positive matrix. In the case $m\geq 0$, Ineq. (29) describes an ellipse $\mathcal{E}$ which is centered at $m_e=\mu L$ with major axis L and minor axis $L^\perp$ of lengths $\sqrt{m/k_1}\geq\sqrt{m/k_2}$, respectively. From Ineq. (28c) we obtain the same equations for the tilded quantities derived from $\tilde{N}$.

The final argument for the derivation of the separability criterion is as follows. By Lemma 6 the state is separable if and only if the three sets described by Ineqs. (28) have a common intersection, i.e. iff $I\equiv\mathcal{E}\cap\tilde{\mathcal{E}}\cap\mathcal{C}\neq\emptyset$. The border of I is contained in the union of the borders of the ellipses and circle: $\partial I\subseteq\partial\mathcal{E}\cup\partial\tilde{\mathcal{E}}\cup\partial\mathcal{C}$. Now we can distinguish two cases, both of which allow to calculate a definite solution to the Ineqs. (28) if the state is separable: Either ∂I has nonempty intersections with the borders of two of the sets $\mathcal{E},\tilde{\mathcal{E}},\mathcal{C}$ or ∂I coincides with the border of one of the three. In the latter case this whole set is contained in I. In the former case, at least one of the points at which the borders intersect must be in I and thus a solution. If no solution is found this way the state is inseparable. This argument is made more precise in the final theorem. Formulas for the nine candidate solutions – the centers $m_c,m_e,m_{\tilde{e}}$ and the intersections points $i^\pm_{e\tilde{e}},i^\pm_{ce},i^\pm_{c\tilde{e}}$ – are given in Appendix A.

Theorem 3 *(Criterion for full separability)*
A three-mode state corresponding to the CM γ *satisfying Ineq. (12) is fully separable if and only if Ineq. (23b) holds and there exists a point* ξ_{sol}

$$\xi_{sol}\in\{m_c,m_e,m_{\tilde{e}},i^\pm_{e\tilde{e}},i^\pm_{ce},i^\pm_{c\tilde{e}}\} \tag{30}$$

fulfilling the Ineqs. (28).

PROOF: We already saw (Obs. 4) that $\det N,\det\tilde{N}>0$ are necessary for separability. If this holds, the quantities used in (28, 30) and in their derivation are all well-defined.

According to Lemma 6 γ is fully separable iff there exists a point $(y,z)^T$ such that the Ineqs. (28) are fulfilled. Therefore, if one of the points (30) satisfies Ineqs. (28) then it determines a γ_A fulfilling Ineqs. (21) thus proving that the state is separable. To complete the proof, we show that if the state is separable, then we find a solution to Ineqs. (28) among the points (30).

As pointed out before, the condition that Ineqs. (28) can simultaneously be satisfied has the geometrical interpretation that the circle $\mathcal{C}$ and the two ellipses $\mathcal{E},\tilde{\mathcal{E}}$ have a nonempty intersection, i.e. $I\equiv\mathcal{E}\cap\tilde{\mathcal{E}}\cap\mathcal{C}\neq\emptyset$.

Thus it remains to prove that if I is nonempty then one of the nine points in (30) lies in I. But if $I\neq\emptyset$ there are only the following two possibilities: since all the sets considered are convex and closed, either the border of I coincides with that of one of the sets $\mathcal{C},\mathcal{E},\tilde{\mathcal{E}}$ (which means that one of these sets, call it $\mathcal{S}$, is contained in *both* others) or at least two of the borders $\partial\mathcal{C},\partial\mathcal{E},\partial\tilde{\mathcal{E}}$ contribute to ∂I, in which case the points at which these two intersect belong to ∂I and thus to I.

In the former case, the center of $\mathcal{S}$ is a solution and given by one of the Eqs. (A1); in the latter, one can find a solution among the intersections of the borders of the sets $\mathcal{E},\tilde{\mathcal{E}},\mathcal{C}$. That these are given by the $i^\pm_x$ is shown in Appendix A. ∎

If a CM γ belongs to a separable state according to the above theorem then the point ξ_{sol} provides us with a pure one-mode CM γ_A such that $N,\tilde{N}\geq\gamma_A$. By construction $\gamma'=B-C(A-\gamma_A)^{-1}C^T$ is a separable 2×2 CM and by repeating a similar procedure as above with γ' we can calculate a pure product-state decomposition of the original state with CM γ.

IV. EXAMPLES OF BOUND ENTANGLED STATES

In this section we construct states belonging to Classes 3 and 4. Our construction makes use of ideas that were first applied in finite dimensional quantum systems to

find PPT entangled states (PPTES) [5] and then generalized in [32] to construct so-called *edge states*, i.e. states on the border of the convex set of states with positive partial transpose. Similarly, one can define "edge CMs" as those that lie on the border of the convex set of PPT CMs (they are called "minimal PPT CMs" in [20]).

This section is divided into three subsections. In the first one we define "edge CMs" and characterize them. In the second and third subsections we present two different families of CMs which contain edge CMs. We also show that within those families we have CMs belonging to all classes.

A. Edge CMs

In the following we will consider CMs γ corresponding to PPT states, i.e. fulfilling

$$\gamma - i\tilde{J}_x \geq 0, \quad \text{for all } x = 0, A, B, C, \tag{31}$$

where $\tilde{J}_0 \equiv J$.

Definition 1 *(Edge Correlation Matrices)* *A CM γ is an edge CM if it corresponds to a non–separable state, fulfills (31), and $\gamma' \equiv \gamma - P$ does not fulfill (31) for all real operators P with $0 \neq P \geq 0$.*

Note that a state with an edge CM automatically belongs to class 4 (i.e. edge CMs correspond to 3-mode biseparable states). In order to fully characterize them, we will need the following definition. Let us consider the complex vector space $V \subseteq \mathbb{C}^6$ of dimension d spanned by the vectors belonging to the kernels of all $\gamma - i\tilde{J}_x$ ($x = 0, A, B, C$). We will define $K(\gamma)$ as a real vector space which is spanned by the real parts and imaginary parts of all the vectors belonging to V. More specifically, let us denote by $B = \{f_R^k + i f_I^k\}_{k=1}^d$ a basis of V, such that f_R^k and f_I^k are real. We define

$$K(\gamma) = \left\{ \sum_k \lambda_k f_R^k + \mu_k f_I^k, \lambda_k, \mu_k \in \mathbb{R} \right\} \subseteq \mathbb{R}^6; \tag{32}$$

that is, the real span of the vectors f_R^k and f_I^k. Note that this definition does not depend on the chosen basis B [As it is pointed out in Appendix B, $K(\gamma)$ coincides with the real vector space spanned by all the vectors in the kernels of $\gamma + \tilde{J}_x \gamma^{-1} \tilde{J}_x$]. We then have the following

Theorem 4 *(Characterization of $1 \times 1 \times 1$ edge CMs)* *A CM γ fulfilling (31) is an edge CM if and only if there exist no CMs $\gamma_A, \gamma_B, \gamma_C$ such that $\gamma = \gamma_A \oplus \gamma_B \oplus \gamma_C$ and $K = \mathbb{R}^6$.*

Proof: We will use the fact [31] that, given two positive matrices $A, B \neq 0$, there exists some $\epsilon > 0$ such that $A - \epsilon B \geq 0$ iff $\text{ran}(B) \subseteq \text{ran}(A)$. According to the Def. 1 we cannot subtract any real positive matrix from γ without violating the conditions (31). This is equivalent to imposing that there is no real vector in the intersection of the ranges of the matrices $\gamma - i\tilde{J}_x$. This is again equivalent to saying that there is no real vector orthogonal to all the $\ker(\gamma - i\tilde{J}_x)$, which in turn is equivalent to $K = \mathbb{R}^6$, since that vector should be orthogonal to all the real and imaginary parts of the vectors spanned by those kernels. Now, if γ corresponds to an entangled state it is clear that $\gamma \neq \gamma_A \oplus \gamma_B \oplus \gamma_C$. Conversely, if $\gamma \neq \gamma_A \oplus \gamma_B \oplus \gamma_C$ was separable, then there must exist some real positive P such that $\gamma - P = \gamma_A \oplus \gamma_B \oplus \gamma_C$ is separable, and therefore fulfills (31), which is not possible. ∎

Note that this Theorem generalizes easily to the cases of more than three parties and more than one mode at each site.

In the construction of the following two examples of tripartite bound entangled states we are going to use this theorem. The idea is to take a CM γ_0 of a pure entangled state [which, of course, does not fulfill (31)] and add real positive matrices until the conditions (31) as well as $K = \mathbb{R}^6$ are fulfilled. If the resulting CM is not of the form $\gamma_A \oplus \gamma_B \oplus \gamma_C$ then Theorem 4 implies that it is an edge CM. In fact, we can add more real positive matrices keeping the state entangled [and fulfilling (31)]. In order to see how much we can add, we can use the criterion derived in the previous section.

This method of constructing CMs belonging to Class 4 also indicates how the corresponding states may be prepared experimentally. Adding a positive matrix P to the CM γ_0 corresponds to the following preparation process: start with an ensemble of states with CM γ_0, displace them randomly by d according to the Gaussian probability distribution with covariance matrix given by the inverse of P. This is a *local* operation (that potentially needs to be supplemented by classical communication) on each individual mode. The state produced by this randomization has CM $\gamma + P$ [20].

B. Example 1

In the first example we start out with an entangled state between the two parties Alice and Bob and the vacuum state in Charlie and add two projectors to the corresponding CM. More specifically, we consider the CMs of the form $\gamma_{a_1,a_2} = \gamma + a_1 P_1 + a_2 P_2$, where

$$\gamma = \gamma_{AB} \oplus \mathbb{1}_C, \tag{33}$$

and

$$\gamma_{AB} = \begin{pmatrix} a & 0 & c & 0 \\ 0 & a & 0 & -c \\ c & 0 & a & 0 \\ 0 & -c & 0 & a \end{pmatrix}, \tag{34}$$

with $a = \sqrt{1 + c^2}$ and c can take any value different from zero. Here, $P_1 = \tilde{p}_1 \tilde{p}_1^T$ and $P_2 = \tilde{p}_1 \tilde{p}_1^T$, where $\tilde{p}_1 = (0, 1, 0, 1, 1, 2)^T$ and $\tilde{p}_2 = (1, 0, -1, 0, 0, 1)^T$.

In order to explain why the CM γ_{a_1,a_2} achieves our purposes, let us first consider the two–mode case in which

the correlation matrix is γ_{AB}. We denote now by $p = p_1 + ip_2$ [where $p_1 = (0,1,0,1)^T$ and $p_2 = (1,0,-1,0)^T$] the eigenvector corresponding to the negative eigenvalue of $\gamma_{AB} - i\tilde{J}_A$ [25]. Since $(-i\tilde{J}_A)^* = -i\tilde{J}_B$ we have that the eigenvector corresponding to the negative eigenvalue of $\gamma_{AB} - i\tilde{J}_B$ is $p^* = p_1 - ip_2$. By adding a sufficiently large multiple of the projectors onto those vectors, we obtain a CM whose partial transposes are positive. Note that in this case (just two modes) this would already make the state separable.

In the case of three modes with a correlation matrix γ the same argumentation applies, namely that by adding some projectors we can make the partial transposes with respect to A and B positive. However, we have to involve C and thereby smear out the initial entanglement between A and B among all three parties. This is exactly what is achieved by adding the projectors P_1 and P_2. If we choose now, for instance, $c = 0.3$, $a_1 = 1$, and $a_2 \approx 0.5531095$, then one can show that the set $K(\gamma_{a_1,a_2})$ defined as in Eq. (32) spans $\mathbb{R}^6$. As mentioned at the end of the previous subsection, since the resulting CM is not of the form $\gamma_A \oplus \gamma_B \oplus \gamma_C$ it corresponds to an edge CM.

In Fig. 1 we illustrate to which class γ_{a_1,a_2} belongs as a function of the parameters $a_{1,2}$. In order to determine this, we have used the criterion derived in the previous section. It is worth noting that γ_{a_1,a_2} never becomes separable. This follows from Theorem 3 and the fact that both $m = \tilde{m} = 0$ for all values of $a_{1,2}$, as can be easily verified. This implies that the two ellipses [c.f. Ineq. (29)] are just two points [which coincide with the centers given in Eq. (A1)]. Thus, the only possibility that the circle and the two ellipses intersect is that the centers of the ellipses are the same and lie inside the circle. It is easy to show that for all values of a_1 and a_2 the centers of the two ellipses are never the same. Thus the state corresponding to the CM γ_{a_1,a_2} is never separable and is a PPTES for all values of a_1, a_2 for which the partial transposes are positive.

C. Example 2

Here we present a family of states which belong either to Class 1, 4, or 5. The states of this family are obtained from a pure GHZ-like state [24] by adding a multiple of the identity, i.e.,

$$\gamma_\alpha = \gamma + \alpha \mathbb{1}, \tag{35}$$

where

$$\gamma = \begin{pmatrix} a & 0 & c & 0 & c & 0 \\ 0 & b & 0 & -c & 0 & -c \\ c & 0 & a & 0 & c & 0 \\ 0 & -c & 0 & b & 0 & -c \\ c & 0 & c & 0 & a & 0 \\ 0 & -c & 0 & -c & 0 & b \end{pmatrix}, \tag{36}$$

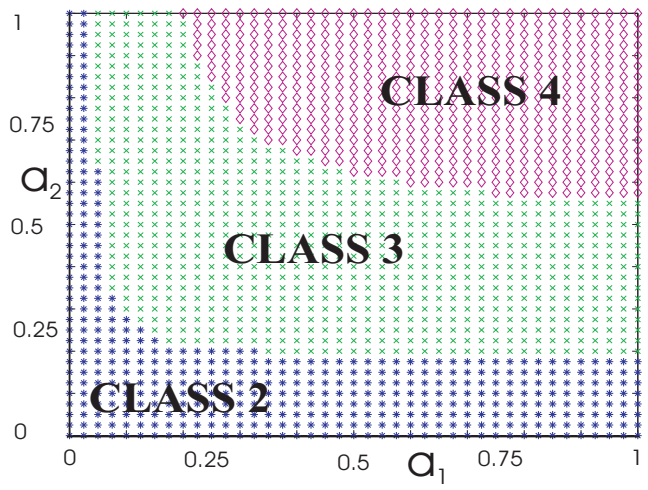

FIG. 1: The Entanglement Classes of γ_{a_1,a_2}

with $a > 1$ and

$$b = \frac{1}{4}(5a - \sqrt{9a^2 - 8}), \tag{37}$$

$$c = \frac{1}{4}(a - \sqrt{9a^2 - 8}). \tag{38}$$

For the following discussion, we pick $a = 1.2$. It is clear that for $\alpha = 0$ the state is fully inseparable, i.e., it belongs to Class 1, whereas for $\alpha \geq 1$ the state will be fully separable (Class 5). We will show now that for $\alpha_0 \leq \alpha \leq \alpha_1$, where $\alpha_0 \approx 0.29756$ and $\alpha_1 \approx 0.31355$ the state is biseparable and belongs therefore to Class 4.

The CM γ_α is symmetric with respect to permutations between the parties, and therefore the negative eigenvalues of the matrices $\gamma - i\tilde{J}_x, x = A, B, C$ are the same. We denote its absolute value by $\alpha_0 \approx 0.29756$. It is easy to determine the real and imaginary part of the corresponding eigenvectors. One finds that all those vectors are linearly independent. If we add now $\alpha_0 \mathbb{1}$ to γ then all those vectors belong to $K(\gamma_{\alpha_0})$ which immediately implies that $K(\gamma_{\alpha_0}) = \mathbb{R}^6$. Since $\gamma_{\alpha_0} \neq \gamma_A \oplus \gamma_B \oplus \gamma_C$ we have that it is an edge CM.

Let us now use Theorem 3 in order to determine α_1. First of all, we show, independently of the discussion above that γ_{α_0} belongs to Class 4. In particular, we find that $m = \tilde{m} = 0$ [cf. Eq. (29)], which implies that there exists a solution to the Ineqs. (28) only if the centers of the two ellipses are the same and lie within the circle. Here one can also show that the two centers are not the same and so the state corresponding to the CM γ_{α_0} is a PPTES. Let us determine the values of α for which it is still the case that there exists no intersection of the two ellipses and the circle given by the Ineqs. (28). It is easy to show that if $\alpha > \alpha_0$ then $\mathrm{tr}N \leq \mathrm{tr}\tilde{N}$, which implies that the circle that has to be considered has radius $r_c = \sqrt{(\mathrm{tr}N)^2/4 - 1}$. One can also easily verify that the two ellipses never intersect the border of the circle,

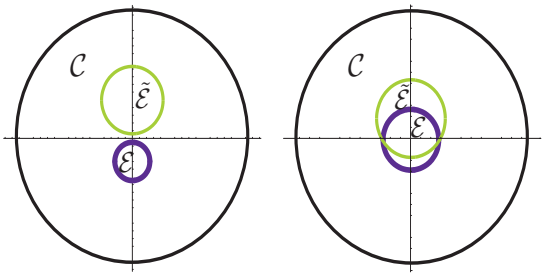

FIG. 2: (a) The circle and the two ellipses do not have a joint intersection, therefore the state corresponding to γ_α is a PPTES, (b) the circle and the two ellipses have a joint intersection, therefore the state corresponding to γ_α is separable.

which simplifies the problem. The ellipses must always lie inside the circle (since if they where outside it would never be possible to obtain a separable state even for $\alpha > 1$). Thus, the problem reduces to check at which point the ellipses intersect each other. This occurs when $\alpha = \alpha_1 \approx 0.31355$. Thus the CM γ_α, where $\alpha_0 \leq \alpha < \alpha_1$ corresponds to a PPTES, whereas for $\alpha \geq \alpha_1$, the corresponding state is fully separable. In Fig. 2 we have plotted the circle and the two ellipses, which are almost circles in this case, for (a) $\alpha < \alpha_1$ and (b) $\alpha > \alpha_1$.

V. CONCLUSIONS

We have discussed nonlocal properties of Gaussian states of three tripartite modes. We have distinguished five classes with different separability properties and given a simple necessary and sufficient criterion that allows to determine which of these classes a given Gaussian state belongs to. The first three classes contain only NPT states and positivity of a state under the three partial transpositions suffices to determine to which of those it belongs. The separability criterion, which allows to distinguish PPT entangled states from separable states is the main result of this paper. For the case of three qubits such a criterion is still missing. Lastly, we have constructed examples for all the classes and in particular for tripartite entangled state with positive partial transpose. Using the separability criterion for multi-mode bipartite Gaussian states [22] the results presented above can be extended to cover the case of n modes at location C. Nothing changes in the argumentation to distinguish 3-party biseparable from fully separable states [the additional modes are taken care of automatically in Eqs. (19)]. However, the separability criterion of [22] is now necessary to determine the properties under bipartite splitting, since for AB-C we deal with a $2 \times n$ state and PPT is then no longer sufficient for biseparability [20].

It is worth pointing out that the separability criterion can be checked experimentally. The CM γ can be measured, and thus the criterion is entirely formulated in terms of quantities that are measurable with current technology.

Gaussian CV states promise to be a fruitful testing ground for quantum nonlocality: Pure entanglement is comparatively easy to create in quantum optical experiments, as described in [24]. Likewise, tripartite bound entangled states are experimentally accessible: the states discussed in the examples Subsec. IV B and IV C can be obtained by mixing differently displaced pure Gaussian states.

The study of entanglement of multi-party Gaussian states is still in a very early stage. For example, no work has to our knowledge been done on the interesting cases of more parties and modes. But even for the simple three-mode case there are important open question. In particular nothing is known about the distillability of tripartite states. As in Ref. [9] for qubits, it is easy to see that Gaussian states in Classes 3 and 4 cannot be distilled at all and are therefore bound entangled. For this, we consider N copies of a Class 3 state ρ, and apply an arbitrary local quantum operation $\mathcal{P}_{locc}$ consisting of a classically correlated sequence of operations of the form $\mathcal{P} = \mathcal{P}_A \otimes \mathcal{P}_B \otimes \mathcal{P}_C$. Since ρ is in Class 3 we can write $\rho^{\otimes N}$ as a mixture of $AB-C$ product states $\sum_k p_k \rho^{(N)}_{AB,k} \otimes \rho^{(N)}_{C,k}$ and as a mixture of $AC-B$ product states $\sum_k p'_k \rho^{(N)}_{AC,k} \otimes \rho^{(N)}_{B,k}$. After applying an operation such as $\mathcal{P}$ the resulting state $\tilde{\rho} = \mathcal{P}(\rho^{\otimes N})$ will still be separable along these cuts, and no sequence of operations $\mathcal{P}$ can change this. Thus ρ is bound entangled.

Whether all states in Class 2 may be distilled to maximally entangled states between the two non-separable parties is an open question. If this were shown, it would follow that all states in Class 1 could be distilled into arbitrary tripartite entangled states.

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APPENDIX A: POINTS OF INTERSECTION

As shown in Theorem 3 a state is separable iff solutions to Ineqs. (28) are found among the points of intersection of the curves described by the *equalities* (28), or the centers of the three sets. Here we give the formulas to directly calculate these points from γ.

The centers of circle and the ellipses have already been shown to be

$$\begin{aligned} m_c &= (0,0)^T, \\ m_e &= \frac{\det N + 1}{k_1} L, \\ m_{\tilde{e}} &= \frac{\det \tilde{N} + 1}{\tilde{k}_1} \tilde{L}, \end{aligned} \tag{A1}$$

where $N, \tilde{N}$ were defined in (19), L in (26), and $k_1, \tilde{k}_1$ after (29). The intersections of the borders of $\mathcal{C}, \mathcal{E}, \tilde{\mathcal{E}}$ are calculated as follows. Consider first the two ellipses, whose borders are defined by the *equalities* (28b, 28c). Dividing by $\mathrm{tr}N$, respectively by $\mathrm{tr}\tilde{N}$ and subtracting the two equalities we find that a point on both $\partial\mathcal{E}$ and $\partial\tilde{\mathcal{E}}$ must lie on the straight line $\mathcal{G}_{e\tilde{e}}$ defined by

$$(\det N + 1 + L^T\xi)/\mathrm{tr}N = (\det \tilde{N} + 1 + \tilde{L}^T\xi)/\mathrm{tr}\tilde{N}, \tag{A2}$$

where $\xi = (y, z)$. $\mathcal{G}_{e\tilde{e}}$ can be parameterized with $s \in \mathbb{R}$ as $g_{e\tilde{e}} + s f_{e\tilde{e}}$, where

$$g_{e\tilde{e}} = \left(\frac{\det N + 1}{\mathrm{tr}N} - \frac{\det \tilde{N} + 1}{\mathrm{tr}\tilde{N}} \right) L'/\|L'\|^2, \tag{A3}$$

where $L' = \tilde{L}/\mathrm{tr}\tilde{N} - L/\mathrm{tr}N$ [33] and $f_{e\tilde{e}}$ is a vector orthogonal to L'.

Inserting $\mathcal{G}_{e\tilde{e}}$ in the equation (28b) for $\partial\mathcal{E}$ we obtain a quadratic polynomial in s, whose roots $s^{\pm}_{e\tilde{e}}$ (if they are real) give the intersection points. For the intersections of $\partial\mathcal{C}$ with the ellipses we proceed similarly. In summary, we get for the intersection points

$$i^{\pm}_{e\tilde{e}} = g_{e\tilde{e}} + s^{\pm}_{e\tilde{e}} f_{e\tilde{e}}, \tag{A4}$$

$$i^{\pm}_{ce} = g_{ce} + s^{\pm}_{ce} f_{ce}, \tag{A5}$$

$$i^{\pm}_{c\tilde{e}} = g_{c\tilde{e}} + s^{\pm}_{c\tilde{e}} f_{c\tilde{e}}, \tag{A6}$$

where the vectors $g_x, x = ce, c\tilde{e}$ are

$$g_{ce} = \left(\mathrm{tr}N\sqrt{r_c^2 + 1} - \det N - 1 \right) L/\|L\|^2, \tag{A7a}$$

f_{ce} is a vector orthogonal to L, and r_c is the smaller of the two radii

$$r_c = \min\left\{ \sqrt{(\mathrm{tr}N)^2/4 - 1}, \sqrt{(\mathrm{tr}\tilde{N})^2/4 - 1} \right\}. \tag{A8}$$

$g_{c\tilde{e}}, f_{e\tilde{e}}$ are defined likewise for tilded quantities. And, finally, by $s^{\pm}_{e\tilde{e}}, s^{\pm}_x$ we denote the real roots of the quadratic polynomials

$$\begin{aligned} P_{e\tilde{e}}(s) &= \left(L^T(g_{e\tilde{e}} + s f_{e\tilde{e}}) + \det N + 1 \right)^2 - \\ &\quad (\mathrm{tr}N)^2 \left(1 + \|g_{e\tilde{e}} + s f_{e\tilde{e}}\|^2 \right), \qquad \text{(A9a)} \\ P_x(s) &= r_c^2 - \|g_x + s f_x\|^2, x = ce, c\tilde{e}. \qquad \text{(A9b)} \end{aligned}$$

Thus all the nine candidates are given in terms of $N, \tilde{N}$ which can be directly obtained from γ.

APPENDIX B: CHARACTERIZATION OF K

Here we show that $K(\gamma)$ as defined in Eq. (32) coincides with the (real) span of the vectors belonging to the kernels of $\gamma + \tilde{J}_x\gamma^{-1}\tilde{J}_x$. This fact automatically follows from the following

Lemma 7 *(Characterization of $K(\gamma)$)*
Let $f = f_R + i f_I$, where f_R and f_I are real. Then $f \in \ker(\gamma - i\tilde{J}_x)$ iff $f_I = \gamma^{-1}\tilde{J}_x f_R$ and both f_R and f_I belong to the kernel of $\gamma + \tilde{J}_x\gamma^{-1}\tilde{J}_x$.

PROOF: Taking the real and imaginary part of the equation $(\gamma - i\tilde{J}_x)f = 0$ we find $\gamma f_R + \tilde{J}_x f_I = 0$ and $\gamma f_I - \tilde{J}_x f_R = 0$. Since γ must be invertible we obtain from the second equation that $f_I = \gamma^{-1}\tilde{J}_x f_R$. Using now the first equation we find that $(\gamma + \tilde{J}_x\gamma^{-1}\tilde{J}_x)f_R = 0$. Analogously, $(\gamma + \tilde{J}_x\gamma^{-1}\tilde{J}_x)f_I = 0$. The same argumentation holds for the other direction of the proof. ∎

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[25] To be precise, we should define J with an index n to keep track of the dimension of the space $\mathbb{R}^{2n}$ on which it acts. But since n will always be clear from the context we will omit this index and just use J to make the expressions more readable.

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[27] A linear transformation S on phase space is called *symplectic* if it preserves J, i.e. if $SJS^T = J$ holds. The symplectic transformations contain those physical operations on CV states that can currently be routinely realized in the lab. They comprise all unitary operations generated by an Hamiltonian quadratic in the canonical operators X_k, P_k, i.e., in quantum optical terms, beam splitter, phase shifter, and squeezer.

[28] In the following, it is convenient to use the notation $A \oplus B$ for block-diagonal matrices: if A and B are $n \times n$ and $m \times m$ square matrices, respectively, then $A \oplus B$ is the $(n+m) \times (n+m)$ square matrix

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

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[30] The following definitions assume that $L, \tilde{L} \neq 0$. (If one of them is 0, the corresponding ellipse degenerates into a circle around $(0,0)$ and we can take an arbitrary $L \neq 0$ to make sense of P_L.) The criterion is not affected by this assumption, since it relies on Ineqs. (28).

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